

A CURVATURE FORMULA FOR THE COMPLEXIFIED INDEX CONE OF A CUBIC FORM

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ABSTRACT. We study the Kähler metric given by the logarithm of a cubic form on its complexified index cone. Under mirror symmetry, this metric should asymptotically correspond to the Weil–Petersson metric. Using the theory of special Kähler manifolds, a proof of a curvature formula for this metric is given.

CONTENTS

1. Introduction	1
2. Affine special geometry	2
3. Projective special geometry	7
4. Complex moduli space as a Special Kähler Manifold	12
5. Complexified index cone of a cubic form as a projective Special Kähler manifold	14
References	16

1. INTRODUCTION

In this paper, we will prove a curvature formula for the Asymptotic Mirror Weil–Petersson metric — or AMWP metric — on the complexified index cone of a cubic polynomial, as studied in [7]. This is motivated by classical mirror symmetry, where on the complexified Kähler cone of a Calabi-Yau threefold we can define a metric corresponding, under the mirror map, to the Weil–Petersson metric on the complex moduli space of a mirror manifold. To obtain the AMWP metric, we neglect the contributions by instanton corrections. A curvature formula for the Weil–Petersson metric was given in [6], where it was also claimed that a similar result should hold on the complexified Kähler cone. Further justification has been given in [7], where the precise conjecture proved in here was made ([7, p.10]) and proved for the special case of ternary cubic polynomials. As the setting in which Strominger proves the curvature formula for the Weil–Petersson metric in [6] is special geometry, we will make use of the same setup here.

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In the first two sections we give the basic definitions of special geometry and state the main results that are needed to prove the curvature formula. In section 4, we quickly recall how special geometry is applied for the complex moduli space of a Calabi–Yau threefold. The proof that the same curvature formula applies to the complexified index cone of a cubic polynomial will be given in section 5. This will include the examples of complexified Kähler cones of the Calabi–Yau threefolds investigated in [7].

2. AFFINE SPECIAL GEOMETRY

In this section, we give the basic definitions of affine special Kähler manifolds and derive a formula for the curvature tensor in terms of the pre-potential. The main reference for this section is Freed’s paper on special Kähler manifolds [4], although we sometimes use slightly different notation and supply proofs where Freed omits them.

Definition 2.1. Let (M, ω) be a Kähler manifold. A *special Kähler structure* on M is a flat, torsion-free connection ∇ on TM such that

$$(2.1) \quad d_{\nabla} I = 0$$

$$(2.2) \quad \nabla \omega = 0$$

Here $d_{\nabla} : \Omega^p(M; TM) \rightarrow \Omega^{p+1}(M; TM)$ denotes the associated exterior covariant derivative.

∇ and d_{∇} give the de Rham complex with coefficients in TM :

$$(2.3) \quad 0 \rightarrow \Omega^0(TM) \xrightarrow{d_{\nabla}} \Omega^1(TM) \xrightarrow{d_{\nabla}} \Omega^2(TM) \xrightarrow{d_{\nabla}} \dots$$

Flatness of ∇ is then expressed by requiring $d_{\nabla}^2 = 0$. It implies that the Poincaré lemma is valid for (2.3).

Lemma 2.2. *We can express the torsion-free condition of ∇ by $d_{\nabla}(\mathbf{id}_{TM}) = 0$, where we identify $\Gamma(\text{End}(TM))$ and $\Omega^1(TM)$.*

Proof. Recall that the torsion tensor is defined as

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

For the exterior covariant derivative we compute

$$\begin{aligned} d_{\nabla}(\mathbf{id}_{TM})(X, Y) &= d_{\nabla}(\mathbf{id}(Y))X - d_{\nabla}(\mathbf{id}(X))Y - \mathbf{id}([X, Y]) \\ &= d_{\nabla}Y(X) - d_{\nabla}X(Y) - [X, Y] \\ &= \nabla_X Y - \nabla_Y X - [X, Y] \end{aligned}$$

□

Now choose a flat local framing $\{e^i\}, i = 1, \dots, n$, for TM with dual framing $\{\theta^i\}, i = 1, \dots, n$, for T^*M .

Lemma 2.3. $d\theta^i = 0$.

Proof. In local coordinates, id_{TM} is given by $\sum_i e^i \otimes \theta^i$. Then $d_\nabla(\text{id}_{TM}) = \sum_i e^i \otimes d\theta^i + \nabla e^i \otimes \theta^i$. As the e^i are a flat TM -framing, i.e. $\nabla e^i = 0$, the result follows. \square

Definition 2.4. We will call a coordinate system $\{x_i, y_j\}, i, j = 1, \dots, n$ *Darboux* if

$$\omega = \sum_k dx_k \wedge dy_k.$$

As our local coframing is exact, we have $\theta_i = dt_i$ for local coordinate functions t_i . Because $\nabla\omega = 0$, we can choose the coordinate functions so that they form a Darboux coordinate system. If we cover M by local Darboux coordinate systems, the transition functions take the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = S \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}, \quad S \in Sp(2n; \mathbb{R}), \quad a, b \in \mathbb{R}^n.$$

As giving a complex structure is equivalent to giving the decomposition of the complexified tangent bundle in holomorphic and anti-holomorphic parts, instead of equation (2.1) we can also require

$$(2.4) \quad d_\nabla \pi^{(1,0)} = 0,$$

where we identify $\pi^{(1,0)} \in \text{Hom}(TM^\mathbb{C}, TM^{1,0})$ with its dual and use

$$\text{Hom}(TM^{1,0}, TM^\mathbb{C}) \cong \Omega^{1,0}(TM^\mathbb{C}).$$

By the Poincaré lemma for (2.3) and $d_\nabla \pi^{(1,0)} = 0$ there exists a complex vector field ξ such that

$$(2.5) \quad \nabla \xi = \pi^{(1,0)}.$$

In local Darboux coordinates,

$$\xi = \frac{1}{2} \sum_{i=1}^n (z_i \frac{\partial}{\partial x_i} - w_i \frac{\partial}{\partial y_i}).$$

and thus

$$\nabla \xi = \pi^{(1,0)} = \frac{1}{2} \sum_{i=1}^n (dz_i \otimes \frac{\partial}{\partial x_i} - dw_i \otimes \frac{\partial}{\partial y_i}).$$

This implies $\text{Re}(dz_i) = dx_i, \text{Re}(dw_i) = -dy_i$, which can be seen by taking the real part of the last equation and using that the identity on TM is the sum of the two projections: $\text{id} = \pi^{1,0} + \pi^{0,1}$. Thus

$$\text{Re}(\pi^{(1,0)}) = \frac{1}{2} \text{id} = \frac{1}{2} \left(\sum_{i=1}^n (dx_i \otimes \frac{\partial}{\partial x_i} + dy_i \otimes \frac{\partial}{\partial y_i}) \right).$$

Definition 2.5. Let (M, ω, ∇) be a special Kähler manifold. A holomorphic coordinate system $\{z_i\}$ is called *special* if $\nabla \text{Re}(dz_i) = 0$. Two special coordinate systems $\{z_i\}, \{w_j\}$ are called *conjugate* if there exists a flat Darboux system $\{x_i, y_j\}$ such that $\text{Re}(dz_i) = dx_i, \text{Re}(dw_j) = -dy_j$.

By [1, p.92], this extension of the Darboux coordinate system to a conjugate pair of special holomorphic coordinate systems is unique up to purely imaginary translations.

The conjugate coordinate systems are related by a change of coordinates via holomorphic functions:

$$(2.6) \quad dw_i = \sum_j \tau_{ij} dz_j$$

Lemma 2.6. [2, p. 757] *The holomorphic 2-form $\Theta = \sum_i dw_i \wedge dz_i$ is everywhere vanishing.*

This implies that $\theta = \sum_i w_i dz_i$ is closed, thus exact.

Definition 2.7. The holomorphic function \mathcal{F} such that $\theta = d\mathcal{F}$ is called the *prepotential*.

Let us choose the prepotential such that the following equations hold:

$$(2.7) \quad w_i = \frac{4\partial\mathcal{F}}{\partial z_i}$$

$$(2.8) \quad \tau_{ij} = \frac{4\partial^2\mathcal{F}}{\partial z_i \partial z_j}$$

Then define $K = \frac{1}{2}Im(w_i \bar{z}_i) = 2Im(\frac{\partial\mathcal{F}}{\partial z_i} \bar{z}_i)$. We compute

$$\begin{aligned} i\partial\bar{\partial}K &= 2i\partial\bar{\partial}Im(\frac{\partial\mathcal{F}}{\partial z_i} \bar{z}_i) \\ &= 2iIm(\frac{\partial^2\mathcal{F}}{\partial z_i \partial z_j}) dz_i \wedge d\bar{z}_j \\ &= -\frac{i}{2}Im(\tau_{ij}) dz_i \wedge d\bar{z}_j \end{aligned}$$

Note that this differs from the conventions in [4], where the Kähler potential is $\frac{1}{2}Im(\frac{\partial\mathcal{F}}{\partial z_i} \bar{z}_i)$.

On our Kähler manifold M we have two affine connections: the Levi-Civita connection D and the flat, torsion-free connection ∇ . Define

$$A_{\mathbb{R}} = \nabla - D; \quad A_{\mathbb{R}} \in \Omega^1(M, End_{\mathbb{R}} TM)$$

Extending D and ∇ to the complexified tangent bundle, we obtain A and \bar{A} .

Theorem 2.8. [2, p. 759, Thm. 3.9]

$$\begin{aligned} A &\in \Omega^{1,0}(Hom(T^{1,0}, T^{0,1})) \\ \bar{A} &\in \Omega^{0,1}(Hom(T^{0,1}, T^{1,0})) \end{aligned}$$

For the curvature of the Kähler metric on M we obtain the following formula:

Theorem 2.9. $R_D = -(A \wedge \bar{A} + \bar{A} \wedge A)$

Proof. By definition, $\nabla = D + A_{\mathbb{R}}$ is flat, so:

$$(2.9) \quad 0 = \nabla^2 = (D + A_{\mathbb{R}})^2 = D^2 + D(A_{\mathbb{R}}) + A_{\mathbb{R}}(D) + A_{\mathbb{R}}^2$$

$$(2.10) \quad = R_D + d_D(A) + A \wedge \bar{A} + \bar{A} \wedge A + A \wedge A + \bar{A} \wedge \bar{A}$$

$$(2.11) \quad = R_D + d_D(A) + A \wedge \bar{A} + \bar{A} \wedge A,$$

as $A \wedge A$ and $\bar{A} \wedge \bar{A}$ vanish. We split the covariant exterior derivative of D : $d_D = \partial_D + \bar{\partial}_D$. As D is the Levi-Civita connection corresponding to ω , we conclude that $\bar{\partial}_D = \bar{\partial}$. By decomposing the equation above into its (r, s) -components, we arrive at:

$$(2.12) \quad 0 = R_D + A \wedge \bar{A} + \bar{A} \wedge A + \partial_D(\bar{A}) + \bar{\partial}(A)$$

We can further split this up into complex linear and anti-linear pieces, which both have to vanish:

$$(2.13) \quad 0 = R_D + A \wedge \bar{A} + \bar{A} \wedge A$$

$$(2.14) \quad 0 = \partial_D(\bar{A}) + \bar{\partial}(A)$$

The complex linear part gives the equation we wanted. Also, in the last equation the two summands take values in different bundles, so we can conclude that A is holomorphic and

$$(2.15) \quad 0 = \partial_D(\bar{A})$$

Thus we arrive at the formula for R we wanted. \square

Definition 2.10. We can also measure the failure of ∇ to preserve the complex structure by the cubic form

$$(2.16) \quad \Xi = -\omega(\pi^{(1,0)}, \nabla \pi^{(1,0)}) = -\omega(\nabla \xi, \nabla^2 \xi),$$

where we use (2.5) for the last equality, which of course only holds locally.

In local coordinates we can then compute that the coefficients of Ξ are given by the third partial derivatives of \mathcal{F} :

Lemma 2.11. $\Xi_{ijk} = \frac{1}{2} \frac{\partial^3 \mathcal{F}}{\partial z_i \partial z_j \partial z_k}$

Proof. We may write $\pi^{(1,0)}$ in local coordinates as

$$\pi^{(1,0)} = \sum_j \frac{\partial}{\partial z_j} \otimes dz_j.$$

Thus, using (2.7) and (2.8) we obtain

$$\begin{aligned}
\Xi &= -\omega \left(\sum_i dz_i \otimes \frac{\partial}{\partial z_i}, \nabla \left(\sum_j dz_j \otimes \frac{\partial}{\partial z_j} \right) \right) \\
&= - \sum_{i,j,k,l,m} \omega \left(\frac{1}{2} \left(\frac{\partial}{\partial x_i} - \tau_{im} \frac{\partial}{\partial y_m} \right), -\frac{1}{2} \frac{\partial \tau_{jl}}{\partial z_k} dz_k \otimes \frac{\partial}{\partial y_l} \right) \otimes dz_i \otimes dz_j \\
&= \frac{1}{8} \sum_{i,j,k} \frac{\partial \tau_{ij}}{\partial z_k} dz_i \otimes dz_j \otimes dz_k \\
&= \frac{1}{2} \sum_{i,j,k} \frac{\partial^3 \mathcal{F}}{\partial z_i \partial z_j \partial z_k} dz_i \otimes dz_j \otimes dz_k.
\end{aligned}$$

□

To compute the components of the curvature tensor in a local coordinate system $\{z_i\}_{i=1,\dots,n}$, we use the formula $0 = R_D + A \wedge \bar{A} + \bar{A} \wedge A$. We have:

$$(2.17) \quad \omega = \sum_{k,j} \frac{i}{2} g_{k\bar{j}} dz_k \wedge d\bar{z}_j$$

$$(2.18) \quad R_D = \sum_{i,j,k,l} (R_{ik\bar{l}}^j)_{i,j} dz_k \wedge d\bar{z}_l$$

$$(2.19) \quad (R_D)_{i\bar{j}k\bar{l}} = \sum_m g_{m\bar{j}} R_{ik\bar{l}}^m$$

$$(2.20) \quad \Xi = \sum_{i,j,k} \Xi_{ijk} dz_i \otimes dz_j \otimes dz_k$$

As $A \in \Omega^{1,0}(Hom(T^{1,0}, T^{0,1}))$, we can expand it in a local framing for the cotangent bundle:

$$(2.21) \quad A = \sum_i A_i dz_i, \quad A_i \in Hom(T^{1,0}, T^{0,1})$$

$$(2.22)$$

We now show how, given a smooth cubic form $\Xi \in C^\infty(M, Sym^3 T^*M)$, we recover the corresponding ∇ , or rather A . In local coordinates ω is given by equation (2.17). Also, the curvature tensor A is locally given by

$$A = \sum_{i,j,k} A_{ij}^{\bar{k}} dz_i \otimes dz_j \otimes \frac{\partial}{\partial z_k}.$$

Then

$$\begin{aligned}
\Xi_{ijk} &= \frac{i}{2} \sum_{p,q} g_{p\bar{q}} dz_p \wedge d\bar{z}_q \left(\frac{\partial}{\partial z_i}, (A_{\frac{\partial}{\partial z_j}} \pi^{(1,0)}) \frac{\partial}{\partial z_k} \right) \\
&= \frac{i}{2} \sum_{p,q} g_{p\bar{q}} dz_p \wedge d\bar{z}_q \left(\frac{\partial}{\partial z_i}, \left(\sum_{l,m} A_{jl}^{\bar{m}} dz_l \otimes \frac{\partial}{\partial \bar{z}_m} \right) \left(\sum_j \frac{\partial}{\partial z_j} \otimes dz_j \right) \frac{\partial}{\partial z_k} \right) \\
&= \frac{i}{2} \sum_{p,q} g_{p\bar{q}} dz_p \wedge d\bar{z}_q \left(\frac{\partial}{\partial z_i}, \left(\sum_{l,m} A_{jl}^{\bar{m}} dz_l \otimes \frac{\partial}{\partial \bar{z}_m} \right) \frac{\partial}{\partial z_k} \right) \\
&= \frac{i}{2} \sum_{p,q} g_{p\bar{q}} dz_p \wedge d\bar{z}_q \left(\frac{\partial}{\partial z_i}, \sum_m A_{jk}^{\bar{m}} \frac{\partial}{\partial \bar{z}_m} \right) \\
&= \frac{i}{2} \sum_q g_{i\bar{q}} A_{kj}^{\bar{q}}.
\end{aligned}$$

Or equivalently

$$A_{jk}^{\bar{q}} = 2i \sum_i g^{\bar{q}i} \Xi_{ijk}.$$

Using (2.13), we obtain the following formula for the curvature of (M, ω) in terms of the cubic form Ξ :

Lemma 2.12.

$$(2.23) \quad R_{i\bar{j}k\bar{l}} = 4 \sum_{p,q} g^{p\bar{q}} \Xi_{ikp} \bar{\Xi}_{jlq}$$

$$(2.24) \quad = \sum_{p,q} g^{p\bar{q}} \mathcal{F}_{ikp} \bar{\mathcal{F}}_{jlq}$$

3. PROJECTIVE SPECIAL GEOMETRY

Projective special Kähler manifolds are basically affine special Kähler manifolds with a \mathbb{C}^* -action. In Freed's setup this is achieved by taking a line bundle L over the Kähler manifold M . Then by deleting the zero section one obtains a principal \mathbb{C}^* -bundle, which naturally equips the total space with a \mathbb{C}^* -action.

Definition 3.1. Let (M, ω) be a Kähler manifold, $L \rightarrow M$ a holomorphic line bundle. (M, L, ω) is called a *Hodge manifold* if $c_1(L) = [\omega]$. This implies that the Kähler class is integral.

As (L, ω_L) is in particular a vector bundle, it has a global zero section. Removing this section, we obtain a principal bundle $\tilde{M} \xrightarrow{\pi} M$ with structure group \mathbb{C}^* . Every holomorphic line bundle admits a hermitian metric, which

implies the existence of the *Chern connection* ∇_L on L , which preserves the hermitian structure of the line bundle and has the property that

$$\nabla_L^{0,1} = \bar{\partial}.$$

∇_L induces a connection on the principal bundle \tilde{M} .

Definition 3.2. Given a vector bundle $\pi : V \rightarrow M$ and a continuous map $f : M' \rightarrow M$, we define the *pullback bundle*

$$f^*V := \{(p, v) \in M' \times V \mid f(p) = \pi(v)\} \subseteq M' \times V$$

together with the natural projection to the first factor: $\pi' : f^*V \rightarrow M'$. For a section $s \in \Gamma(M, V)$ we define a *pullback section* $f^*s \in \Gamma(M', V)$ by $f^*s = s \circ f$.

The pullback bundle $\pi^*L \rightarrow \tilde{M}$ has a canonical nonzero holomorphic section s .

We define a pseudo-Kähler metric on \tilde{M} :

Definition 3.3. Let $\tilde{\omega} \in \Omega^{1,1}(\tilde{M})$ denote the form which is given by:

- (1) $\tilde{\omega}(U, X) = 0$
- (2) $\tilde{\omega}(X, Y) = |s|^2 \pi^* \omega(X, Y)$
- (3) $\tilde{\omega}(U, V) = -\frac{i}{2\pi} \partial \bar{\partial} |s|^2(U, V)$,

where X, Y are horizontal vector fields and U, V are vertical vector fields.

In the following we will establish a curvature formula for $\tilde{\omega}$ in terms of ω and the cubic form Ξ on M .

Lemma 3.4. $(\tilde{M}, \tilde{\omega})$ is a pseudo-Kähler manifold. Furthermore

$$\tilde{\omega} = \frac{i}{2\pi} \partial \bar{\partial} |s|^2$$

and

$$\pi^* \omega = \frac{i}{2\pi} \partial \bar{\partial} \log |s|^2$$

Proof. The horizontal lift of a vector field $X \in TU$ is $X - \lambda h^{-1} \partial h(X) \frac{\partial}{\partial \lambda}$. Also, as by this formula $\tilde{\omega}$ is exact and real of type $(1, 1)$, it defines a pseudo-Kähler metric of signature $(\dim(M), 1)$ on \tilde{M} . It will not define a Kähler metric as it will be negative on vertical vector fields. The second formula is just the standard curvature formula for the Chern connection of a holomorphic line bundle, up to the constant factor of $\frac{i}{2\pi}$. \square

Using this characterisation of the metrics, we can compute the Kähler metric \tilde{g} on \tilde{M} in terms of the Kähler metric g on M . We choose local coordinates z_i on $U \subset M$ and a coordinate λ on $L|_U$. Choosing a nonzero local section t of $L|_U$, we get local coordinates on $\pi^*(U) \subset \tilde{M}$ by $(z, \lambda) \mapsto \lambda t(z), \lambda \in \mathbb{C}^*$.

Lemma 3.5. *Let g, \tilde{g} be the Kähler metric on M and \tilde{M} . Let $K_i := \partial_i \log K$, K the Kähler potential of $\tilde{\omega}$. Also, we use the index 0 in the metric for the fibre coordinate. Then*

$$(3.1) \quad \tilde{g}_{i\bar{j}} = K(-g_{i\bar{j}} + K_i K_{\bar{j}})$$

$$(3.2) \quad \tilde{g}_{0\bar{i}} = K \lambda^{-1} K_i$$

$$(3.3) \quad \tilde{g}_{0\bar{0}} = (\lambda \bar{\lambda})^{-1} K$$

$$(3.4) \quad \tilde{g}^{i\bar{j}} = -K^{-1} g^{i\bar{j}}$$

$$(3.5) \quad \tilde{g}^{0\bar{i}} = \bar{\lambda} K^{-1} \sum_{k=1}^n g^{k\bar{i}} K_{\bar{k}}$$

$$(3.6) \quad \tilde{g}^{0\bar{0}} = K^{-1} \lambda \bar{\lambda} (1 - \sum_{i,j=1}^n K_i K_{\bar{j}} g^{j\bar{i}})$$

The entries of the inverse metric have been found by fixing the $\tilde{g}^{i\bar{j}}$ for $i, j = 1, \dots, n$ to give the right answer when computing the Strominger formula in the projective case. This determines the other entries and indeed gives an inverse matrix to \tilde{g} .

A tedious but easy calculation leads to expressions for the connection coefficients of the Levi-Civita connection of \tilde{g} in terms of the coefficients of the Levi-Civita connection for g . As both metrics are Kähler, we only have to compute the ones which might be non-vanishing:

Lemma 3.6. *With notation as in the previous lemma, we compute the connection coefficients $\tilde{\Gamma}_{jk}^i$, using $\Gamma_{jk}^i = g^{i\bar{l}} \partial_j g_{\bar{l}k}$:*

$$(3.7) \quad \tilde{\Gamma}_{jk}^i = \Gamma_{jk}^i + K_j \delta_k^i + K_k \delta_j^i$$

$$(3.8) \quad \tilde{\Gamma}_{j0}^i = \lambda^{-1} \delta_j^i$$

$$(3.9) \quad \tilde{\Gamma}_{ij}^0 = \lambda \Gamma_{ij}^k K_k + 2\lambda K_i K_j - K^{-1} \lambda \partial_i \partial_j K$$

$$(3.10) \quad \tilde{\Gamma}_{00}^0 = \tilde{\Gamma}_{i0}^0 = \tilde{\Gamma}_{00}^i = 0$$

It immediately follows that

$$(3.11) \quad \Gamma_{jk}^i = \tilde{\Gamma}_{jk}^i - \lambda \tilde{\Gamma}_{j0}^i K_k - \lambda \tilde{\Gamma}_{0k}^i K_j$$

We can now compute the $(3, 1)$ curvature tensor $R_{ik\bar{l}}^m$ of the Levi-Civita connection on M , using the following formula:

$$R_{ik\bar{l}}^m = -\partial_{\bar{l}} \Gamma_{ik}^m$$

Expressed in terms of the $(3, 1)$ curvature tensor $\tilde{R}_{ik\bar{l}}^m$ on \tilde{M} and the Kähler potential K , we obtain:

$$R_{ik\bar{l}}^m = -(\tilde{R}_{ik\bar{l}}^m - \partial_{\bar{l}}(\delta_i^m K_k + \delta_k^m K_i)),$$

where $i, j, k, m \in \{1, \dots, n\}$. Expressing $\tilde{R}_{ik\bar{l}}^m$ in terms of the cubic form, we obtain:

$$R_{ik\bar{l}}^m = -4 \sum_{p,q,m,n} \tilde{g}^{p\bar{q}} \tilde{g}^{n\bar{m}} \tilde{\Xi}_{ikp} \tilde{\Xi}_{mlq} + \partial_{\bar{l}}(\delta_i^m K_k + \delta_k^m K_i)$$

Plugging in (2.23), we obtain for the $(4, 0)$ curvature tensor:

$$\begin{aligned} R_{i\bar{j}k\bar{l}} &= \sum_m g_{m\bar{j}} R_{ik\bar{l}}^m \\ &= \sum_m g_{m\bar{j}} (-\tilde{R}_{ik\bar{l}}^m + \partial_{\bar{l}}(\delta_i^m K_k + \delta_k^m K_i)) \\ &= -\sum_m g_{m\bar{j}} 4 \sum_{p,q,n} \frac{1}{K} g^{p\bar{q}} \frac{1}{K} g^{n\bar{m}} \tilde{\Xi}_{ikp} \tilde{\Xi}_{mlq} + \sum_m g_{m\bar{j}} \partial_{\bar{l}} \delta_i^m K_k + \sum_m g_{m\bar{j}} \partial_{\bar{l}} \delta_k^m K_i \\ &= -\frac{4}{K^2} \sum_{p,q} g^{p\bar{q}} \Xi_{ikp} \bar{\Xi}_{jlq} + g_{i\bar{j}} \partial_{\bar{l}} K_k + g_{k\bar{j}} \partial_{\bar{l}} K_i \\ &= -\frac{4}{K^2} \sum_{p,q} g^{p\bar{q}} \Xi_{ikp} \bar{\Xi}_{jlq} + g_{i\bar{j}} g_{k\bar{l}} + g_{k\bar{j}} g_{i\bar{l}} \\ &= -\frac{1}{K^2} \sum_{p,q} g^{p\bar{q}} \mathcal{F}_{ikp} \bar{\mathcal{F}}_{jlq} + g_{i\bar{j}} g_{k\bar{l}} + g_{k\bar{j}} g_{i\bar{l}} \end{aligned}$$

For the last equality we take the above description of $\tilde{\omega}$. Using this in definition 2.10 and remembering that $\tilde{\nabla}$ is just the pullback of ∇ by the \mathbb{C}^* -action, we see that $\tilde{\Xi}$ and Ξ do not differ for the vector fields coming from M . It should be noted that for a vector field X on M we have to use the horizontal lift to \tilde{M} to make this identification. The final step is to rewrite the Kähler potential in terms of the cubic form:

Proposition 3.7. *Denote the coordinates on \tilde{M} by λ, z_1, \dots, z_n with $z_j = x_j + iy_j$. In special coordinates $t_j = z_j/\lambda, j = 1, \dots, n$, writing ∂_j for $\frac{\partial}{\partial t_j}$, the Kähler potential for the Weil–Petersson metric is given by*

$$K = -\log i \left(\sum_j ((t_j - \bar{t}_j)(\partial_j \mathcal{F}' + \bar{\partial}_j \bar{\mathcal{F}}')) + 2\bar{\mathcal{F}}' - 2\mathcal{F}' \right) = 8\mathcal{F}(y_1, \dots, y_n)$$

Proof. Let the $t_j = \frac{z_j}{\lambda}$ form a local coordinate system and define $\mathcal{F}'(t) = \lambda^{-2} \mathcal{F}(z)$. Then observe that $\lambda \partial_j \mathcal{F}' = \frac{\partial \mathcal{F}}{\partial z_j}$. Going to inhomogeneous coordinates, we have to account for the following term that has to be corrected in the sum:

$$\lambda \frac{\partial \mathcal{F}}{\partial \lambda} = \lambda^2 (2\mathcal{F}' - \sum_j t_j \partial_j \mathcal{F}')$$

Plugging this into the formula given in the Lemma above we obtain the result. \square

Definition 3.8. [4, p.46] Let (M, L, ω) be an n -dimensional Hodge manifold. A *projective special Kähler structure* is a triple (V, ∇, Q) , where

- (1) $V \rightarrow M$ is a rank $n+1$ holomorphic vector bundle with a holomorphic inclusion $L \hookrightarrow V$.
- (2) ∇ is a flat connection on $V_{\mathbb{R}}$, the underlying real vector bundle, such that the extension of ∇ to the complexification $(V_{\mathbb{R}})_{\mathbb{C}}$ satisfies $\nabla(L) \subset V$ and the section

$$\begin{aligned} M &\rightarrow \mathbb{P}[(V_{\mathbb{R}})_{\mathbb{C}}] \\ m &\mapsto L_m \end{aligned}$$

is an immersion with respect to ∇ , or equivalently, L , seen as a section of $\mathbb{P}[(V_{\mathbb{R}})_{\mathbb{C}}]$, is transverse to the horizontal distribution of the connection.

- (3) Q is a non-degenerate skew-symmetric form on $V_{\mathbb{R}}$ of type $(1, 1)$ with respect to the complex structure. Moreover, it should be flat with respect to ∇ . Finally, taking the extension to $(V_{\mathbb{R}})_{\mathbb{C}}$, we assume $Q|_{L \times \bar{L}} = \frac{i}{2\pi} \omega_L$.

Remark 3.9. By $\nabla(L) \subset V$ we mean that if we take a section $s \in \Gamma(L)$, where L is embedded in V , we get $\nabla(s) \in \Gamma(V) \otimes \Omega_M$.

A projective special Kähler structure defines a certain kind of variation of Hodge structures. If we furthermore require the existence of a lattice in $V_{\mathbb{R}}$ such that Q restricted to the lattice takes integer values, the corresponding variation of Hodge structures will be polarised.

The connection to affine special geometry is established by the following proposition:

Proposition 3.10. [4, p.46] *Let (M, L, ω) be a Hodge manifold with associated pseudo-Kähler manifold $(\tilde{M}, \tilde{\omega})$ and canonical holomorphic non-zero section s . Then a projective special Kähler structure on (M, L, ω) is equivalent to a \mathbb{C}^* -invariant special pseudo-Kähler structure $\tilde{\nabla}$ on $(\tilde{M}, \tilde{\omega})$ with $\tilde{\nabla}s = \pi^{(1,0)}$.*

The condition $\tilde{\nabla}s = \pi^{(1,0)}$ is needed to guarantee that the map in part (2) of the definition of a projective Kähler structure is an immersion: consider s as a holomorphic vertical vector field on \tilde{M} and denote by $\tilde{\nabla}$ the pullback of the connection to π^*V . Then the map

$$(3.12) \quad \begin{aligned} \tilde{\nabla}s : T\tilde{M} &\rightarrow \pi^*V \\ \partial/\partial z_i &\mapsto \nabla_{\partial/\partial z_i} s \end{aligned}$$

is an isomorphism if and only if s is transverse to the horizontal distribution of the connection. The underlying real isomorphism induces a real flat connection on $T\tilde{M}$.

If we denote the sections of V by s, l_1, \dots, l_n , the immersion condition says that the effect of the connection on s is given by:

$$\tilde{\nabla}s = \sum_{j=0}^n \frac{\partial}{\partial z_j} \otimes dz_j$$

This implies that under (3.12), s corresponds to a complex vector ζ field that satisfies $\tilde{\nabla}\zeta = \pi^{(1,0)}$ because of $\pi^{(1,0)} = \sum_j \frac{\partial}{\partial z_j} \otimes dz_j$.

Another useful fact is that \tilde{Q} , the pullback of the skew-symmetric form via π , pulls back to $-\tilde{\omega}$ under the isomorphism, which can be seen by differentiating the equation we assume in point (iii) of the projective special Kähler structure. Using this together with definition 2.10 and the fact that $\omega(\zeta, \tilde{\nabla}\zeta) = 0$ as ζ is a holomorphic $(1,0)$ vector field, we conclude that the cubic form is given by

$$(3.13) \quad \Xi = -\omega(\zeta, \tilde{\nabla}^3\zeta) = \tilde{Q}(\tilde{\nabla}^3s, s).$$

In the following sections we will describe two instances of projective special Kähler manifolds that arise in Mirror Symmetry, the complex moduli space and the (complexified) Kähler moduli space of a Calabi–Yau threefold. A curvature formula for the former can be found in [6], although it is derived in a slightly different fashion. We want to compute the curvature of the complexified Kähler moduli space of a Calabi–Yau threefold with the line bundle $\mathcal{H}^{0,0}$. We do that by proving the formula for the more general case of a complexified index cone of a cubic form.

4. COMPLEX MODULI SPACE AS A SPECIAL KÄHLER MANIFOLD

Let X be a Calabi–Yau threefold, \mathcal{M} the moduli space of complex structures. Over \mathcal{M} , consider the Hodge bundle \mathcal{H} with fibre $H^3(X, \mathbb{C})$ and Hodge filtration \mathcal{F}^\bullet . This gives a $2n+2$ -dimensional vector bundle over \mathcal{M} . Using integral cohomology, we can equip it with a flat connection, called the *Gauss–Manin connection*. There exists a symplectic basis for $H^3(X, \mathbb{C})$ and this provides local flat sections for \mathcal{M} with respect to the Gauss–Manin connection on \mathcal{H} . These bases are unique up to $Sp(2n+2, \mathbb{R})$ transformations. Thus \mathcal{H} is a flat, holomorphic $Sp(2n+2, \mathbb{R})$ bundle.

As all the information about the Hodge structure is already contained in $H^{3,0} \oplus H^{2,1}$, we get an $n+1$ -dimensional sub-bundle $\mathcal{V} \cong \mathcal{F}^2$. On \mathcal{H} , we have two metrics: One is a hermitian metric given by the polarisation of Hodge structure $Q(C', \cdot)$ (in most accounts called Hodge metric), where C is the Weil operator, i.e. acting by multiplication by $i^{p-(n-p)}$ on $H^{p,q}$. The other one is given just by taking the polarisation with a leading factor of $-i^n$, which in general is not positive definite. We will call it the *indefinite metric*. These two metrics differ by $(-1)^p$ on $H^{p,q}$. While the indefinite metric is

by definition preserved by the Gauss–Manin-connection, the Hodge metric is not. This implies that ∇^{GM} is not the Chern connection of the hermitian line bundle $\mathcal{H}^{3,0}$.

On any Calabi–Yau manifold of the family described by the complex moduli space, the nowhere-vanishing holomorphic 3-form Ω is only defined up to multiplication by a non-zero complex number. So in this family, we can multiply Ω by a non-vanishing holomorphic function $f(z)$. As we want no-where vanishing 3-forms, we take the following transformation for Ω :

$$\Omega' = e^f(z)\Omega,$$

where f is a continuous function of the moduli space coordinates. Now at different points in the complex moduli space, we have different complex structures and thus a different decomposition of the tangent bundle of our Calabi–Yau threefold in holomorphic and anti-holomorphic parts. This means that $(3,0)$ -forms that are holomorphic in one complex structure may not stay holomorphic in another complex structure. As Ω is by definition a holomorphic form, it defines us a section of H^3 . The multiples of Ω define the line bundle $\mathcal{H}^{3,0}$ which is a sub-bundle of both \mathcal{H} and \mathcal{V} . We also note for later that because of the freedom in choice for Ω we obtain a section of the projectivization of \mathcal{V} .

We equip \mathcal{M} with a projective special Kähler structure (\mathcal{V}, ∇, Q) given by:

- (1) The vector bundle $\mathcal{V} := \mathcal{H}^{3,0} \oplus \mathcal{H}^{2,1}$ with the holomorphic inclusion $L := \mathcal{H}^{3,0} \hookrightarrow \mathcal{H}^{3,0} \oplus \mathcal{H}^{2,1}$.
- (2) The Gauss–Manin connection ∇^{GM} on the underlying real bundle $\mathcal{V}_{\mathbb{R}}$. For the rest of this chapter, we will denote the Gauss–Manin connection simply by ∇ .
- (3) The non-degenerate form Q is the polarisation of the variation of Hodge structures on the complex moduli space, divided by 2π . Then Q is flat with respect to ∇^{GM} .

Lemma 4.1. (\mathcal{V}, ∇, Q) defines a projective special Kähler structure.

Proof. The holomorphicity of the inclusion $L \hookrightarrow \mathcal{V}$ follows from a theorem of Griffiths [5, p.24] stating that the bundles in the Hodge filtration \mathcal{F}^p corresponding to \mathcal{H} are holomorphic sub-bundles of the full Hodge bundle \mathcal{H} . Here $\mathcal{F}^p = \sum_{i=p}^3 \mathcal{H}^{i,3-i}$. The requirement that $\nabla(L) \subset \mathcal{V}$ after complexification of the underlying real bundles is just Griffiths transversality for the Hodge bundles. For the immersion condition in part (2) of the definition of a projective special Kähler structure, we note that \mathcal{V} is generated by Ω and $\nabla_{\frac{\partial}{\partial z_i}} \Omega$, $i = 1, \dots, n$. But L is generated by the non-vanishing $(3,0)$ -form Ω and $\nabla_{\frac{\partial}{\partial z_i}} \Omega$ is a non-zero section of \mathcal{V} , so L , seen as a section of $\mathbb{P}[(\mathcal{V}_{\mathbb{R}})_{\mathbb{C}}]$, is transverse to the horizontal distribution of the connection. The remaining required properties for Q are immediate from the definition. \square

With this setup and the calculations of the previous section, we obtain the following curvature formula for the complex moduli space appearing in Strominger's paper [6]:

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}} - e^{2K} \sum_{p,q} g^{p\bar{q}} Y_{ikp} \overline{Y_{jlq}},$$

where g is the Weil–Petersson metric, given by taking as Kähler potential $\log Q(\Omega, \bar{\Omega})$, and Y_{ijk} denotes the Yukawa couplings (see [3, p.102] for details). We use (3.13) to obtain as the cubic form Ξ the well-known formula for the Yukawa couplings. The cubic form does not fix the prepotential completely, but as the curvature only depends on Ξ , we obtain the well-known original formula for the complex moduli space.

Proofs of this formula for the Weil–Petersson metric on the complex moduli space of a Calabi–Yau threefold have appeared in the literature before. A proof relying on the earlier extrinsic definition of special geometry via special coordinates and their behaviour under coordinate changes can be found in [6]. Two more proofs, using different methods — most importantly a reformulation of the problem in terms of the period mapping and a theorem by Griffiths on the curvature of Hodge bundles — have been found by Wang [8].

5. COMPLEXIFIED INDEX CONE OF A CUBIC FORM AS A PROJECTIVE SPECIAL KÄHLER MANIFOLD

Take any cubic polynomial with real coefficients $f(y_1, \dots, y_n)$ defined on \mathbb{R}^n .

Definition 5.1. The *index cone* of f is the open cone $W \subset \mathbb{R}^n$ where f is positive and the Hessian matrix $(\partial^2 f / \partial y_i \partial y_j)$ has index $(1, n-1)$.

Definition 5.2. The *complexified index cone* M is given by $(\mathbb{R}^n + iW) / \text{im}(\mathbb{Z}^n)$, where \mathbb{Z}^n is mapped into the first summand.

Let $t_i = x_i + iy_i$. As was shown in [7, p.7ff.], starting with the complexification of a cubic polynomial f — e.g. the cubic intersection form of a Calabi–Yau threefold — we get a Kähler potential from it by using the formula

$$K = -\log i \left(\sum_j (t_j - \bar{t}_j) (\partial_j f + \bar{\partial}_j \bar{f}) + 2\bar{f} - 2f \right).$$

This potential function is independent of the x_i , and is just a multiple of the original polynomial f .

We consider the Hessian metric $(-\frac{1}{4} \partial^2 \log f / \partial y_i \partial y_j)$ on the complexified index cone, obtained by taking K as the Kähler potential. By [7, Lemma 2.1], this defines a Kähler metric on the whole complexified index cone.

It was conjectured in [7, p.10] that the following equivalent of the Strominger formula holds for the Hessian metric on the complexified index cone:

$$(5.1) \quad R_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}} - \sum_{p,q} g^{p\bar{q}} \frac{f_{ikp}f_{jlq}}{64f^2}$$

The factor of 64 is due to the fact that $K = 8\mathcal{F}(y) = 8f$, so $\frac{1}{K^2} = \frac{1}{64f^2}$.

In the following we will show that the complexified index cone together with the Hessian metric given by the cubic polynomial f and the trivial complex line bundle $L = \mathbb{C} \times M$ form a projective special Kähler manifold. Denote the total space obtained by deleting the zero section from L again by \tilde{M} . We choose local coordinates z_1, \dots, z_n, λ , where λ denotes the fibre coordinate.

Then a projective special Kähler structure (\mathcal{V}, ∇, Q) on \tilde{M} given by:

- (1) The vector bundle $\mathcal{V} := \mathbb{C}^{n+1}$ with the obvious holomorphic inclusion of $L = \mathbb{C}$. The underlying real bundle is just \mathbb{R}^{n+1} .
- (2) Choose rational non-vanishing sections s, s_1, \dots, s_n for V , with s the section for L . Our aim is to identify these sections with the basic vector fields $\partial/\partial z_i, \partial/\partial \lambda$, where the former should be identified with s_1 up to s_n and the last one should be identified with s . We use the shorthand notation ∂z for the vector field $\partial/\partial z$. On the real bundle define a connection by setting

$$\begin{aligned} \nabla_{\partial z_i} s &= s_i \\ \nabla_{\partial z_i} s_j &= \sum_{k=1}^n \frac{\partial^3 f}{\partial z_i \partial z_j \partial z_k} \bar{s}_k \\ \nabla_{\partial z_i} \bar{s}_j &= \delta_j^i \bar{s} \\ \nabla_{\partial z_i} \bar{s} &= 0, \end{aligned}$$

where \bar{s}_k are rational sections of $(\mathbb{C}^n)^\vee$, the dual to $V \setminus L$.

- (3) The non-degenerate form Q is given by $Q(\alpha, \beta) = \frac{1}{2}\alpha\beta$.

We compute

$$\begin{aligned} \Xi_{ijk} &= Q(\tilde{\nabla}_{\partial z_i} \tilde{\nabla}_{\partial z_j} \tilde{\nabla}_{\partial z_k} s, s) \\ &= Q(\tilde{\nabla}_{\partial z_i} \tilde{\nabla}_{\partial z_j} s_k, s) \\ &= Q(\tilde{\nabla}_{\partial z_i} \sum_{l=1}^n \frac{\partial^3 f}{\partial z_i \partial z_j \partial z_l} \bar{s}_l, s) \\ &= Q(\frac{\partial^3 f}{\partial z_i \partial z_j \partial z_k} \bar{s}, s) \\ &= \frac{1}{2} \frac{\partial^3 f}{\partial z_i \partial z_j \partial z_k} \end{aligned}$$

which shows that (5.1) holds.

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